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(Extensions of the historical calculus transforms in the geometric function theory)

AUTHOR(S):

Nunokawa, Mamoru; Owa, Shigeyoshi; Shiraishi, Hitoshi

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On a remark of strongly convex functions of order β and convex of order α

Mamoru Nunokawa, Shigeyoshi Owa and Hitoshi Shiraishi

Abstract

For analytic functions $f(z)$ in the open unit disk \mathbb{U} , two subclasses $\mathcal{C}(\alpha, \beta)$ and $\mathcal{S}^*(\alpha, \beta)$ are introduced. The object of the present paper is to investigate a strongly starlikeness of order δ and starlikeness of order $\beta(\alpha)$ of strongly convex functions of order γ and convex of order α .

1 Introduction

Let \mathcal{S} denote the set of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

Suppose that $f(z) \in \mathcal{S}$ satisfies the following conditions

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}) \quad (1)$$

or

$$\left| \arg \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}), \quad (2)$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. If $f(z) \in \mathcal{S}$ satisfies (1), then we say that $f(z)$ is strongly convex of order β and convex of order α in \mathbb{U} and we denote by $\mathcal{C}(\alpha, \beta)$ the class of such functions $f(z)$. If $f(z) \in \mathcal{S}$ satisfies (2), then $f(z)$ is said to be strongly starlike of order β and starlike of order α in \mathbb{U} and we also denote by $\mathcal{S}^*(\alpha, \beta)$ the class of such functions $f(z)$.

In view of the results by MacGregor [1] and Wilken and Feng [3], it is well known that $f(z) \in \mathcal{C}(\alpha, 1)$ implies $f(z) \in \mathcal{S}^*(\beta(\alpha), 1)$ where

$$\beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & (0 \leq \alpha < 1; \alpha \neq \frac{1}{2}) \\ \frac{1}{2\log 2} & (\alpha = \frac{1}{2}). \end{cases}$$

In the present paper, we discuss some properties for $f(z)$ concerning with the classes $\mathcal{C}(\alpha, \beta)$ and $\mathcal{S}^*(\alpha, \beta)$.

2 Main result

To consider our problems, we have to recall here the following lemma due to Nunokawa [2].

Lemma 1. *Let $p(z)$ be analytic in \mathbb{U} with $p(0) = 1$ and $p(z) \neq 0$ in \mathbb{U} . Suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

where $\alpha > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{where} \quad \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{where} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where

$$p(z_0)^{1/\alpha} = \pm ia \quad (a > 0).$$

Now, we derive

Theorem 1. *If $f(z) \in \mathcal{C}(\gamma, \alpha)$ with $0 \leq \alpha < 1$, then $f(z) \in \mathcal{S}^*(\delta, \beta(\alpha))$, where $0 < \delta < 1$ and*

$$(i) \quad \text{if } k = \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1 \text{ and}$$

$$\tan^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} \geq \frac{\pi}{2}\delta,$$

then we put

$$\begin{aligned}\gamma &= \delta + \frac{2}{\pi} \text{Tan}^{-1} \frac{R_0 \sin(\Theta - \frac{\pi}{2}\delta)}{1 + R_0 \cos(\Theta - \frac{\pi}{2}\delta)}, \\ \Theta &= \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}, \\ R_0 &= \frac{\delta}{2(1 - \beta(\alpha)^2)} \left\{ \left(\frac{1 + \delta}{1 - \delta} \right)^{\frac{1-\delta}{2}} + \left(\frac{1 - \delta}{1 + \delta} \right)^{\frac{1+\delta}{2}} \right\},\end{aligned}$$

(ii) if $k = \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1$ and

$$\text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} < \frac{\pi}{2} \delta,$$

then we put

$$\gamma = \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)},$$

(iii) if $k = \frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1$ and

$$\text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} \leq -\frac{\pi}{2} \delta,$$

then we put

$$\begin{aligned}\gamma &= -\delta - \frac{2}{\pi} \text{Tan}^{-1} \frac{R_0 \sin(\Theta + \frac{\pi}{2}\delta)}{1 + R_0 \cos(\Theta + \frac{\pi}{2}\delta)}, \\ \Theta &= \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}, \\ R_0 &= \frac{\delta}{2(1 - \beta(\alpha)^2)} \left\{ \left(\frac{1 + \delta}{1 - \delta} \right)^{\frac{1-\delta}{2}} + \left(\frac{1 - \delta}{1 + \delta} \right)^{\frac{1+\delta}{2}} \right\}\end{aligned}$$

(iv) if $k = \frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1$ and

$$\text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)} > -\frac{\pi}{2} \delta,$$

then we put

$$\gamma = \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}.$$

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Then we have that $p(z)$ is analytic in \mathbb{U} and $p(0) = 1$. It follows that

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)},$$

and

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} - \alpha &= p(z) - \beta(\alpha) + \frac{zp'(z)}{p(z)} + \beta(\alpha) - \alpha \\ &= (p(z) - \beta(\alpha)) \left(1 + \frac{z(p(z) - \beta(\alpha))'}{p(z) - \beta(\alpha)} \frac{1}{p(z)} + \frac{\beta(\alpha) - \alpha}{p(z) - \beta(\alpha)} \right). \end{aligned}$$

Therefore, we see that

$$\begin{aligned} &\arg \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \\ &= \arg(p(z) - \beta(\alpha)) + \arg \left(1 + \frac{z(p(z) - \beta(\alpha))'}{p(z) - \beta(\alpha)} \frac{1}{p(z)} + \frac{\beta(\alpha) - \alpha}{p(z) - \beta(\alpha)} \right). \end{aligned}$$

If there exists a point z_0 , $|z_0| < 1$ such that

$$|\arg(p(z) - \beta(\alpha))| < \frac{\pi}{2}\delta \quad (|z| < |z_0|)$$

and

$$|\arg(p(z_0) - \beta(\alpha))| = \frac{\pi}{2}\delta,$$

then, let us put

$$q(z) = \frac{p(z) - \beta(\alpha)}{1 - \beta(\alpha)}, \quad q(0) = 1.$$

Applying Lemma 1, we have that

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{z_0 q'(z_0)}{p(z_0) - \beta(\alpha)} = i\delta k$$

where

$$q(z_0)^{\frac{1}{\delta}} = \left(\frac{p(z_0) - \beta(\alpha)}{1 - \beta(\alpha)} \right)^{\frac{1}{\delta}} = \pm ia \quad (a > 0)$$

and

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1 \quad \text{when } \arg(p(z) - \beta(\alpha)) = \frac{\pi}{2}\delta$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1 \quad \text{when } \arg(p(z) - \beta(\alpha)) = -\frac{\pi}{2}\delta.$$

At first, let us consider the case $\arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2}\delta$, then it follows that

$$\begin{aligned}
& \arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right) \\
&= \arg(p(z_0) - \beta(\alpha)) \left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \frac{1}{p(z_0)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)}\right) \\
&= \frac{\pi}{2}\delta + \arg\left(1 + \frac{i\delta k}{(\beta(\alpha) + (ia)^\delta)(1 - \beta(\alpha))} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^\delta}\right) \\
&\geq \frac{\pi}{2}\delta + \arg\left(1 + \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))(ia)^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^\delta}\right) \\
&= \frac{\pi}{2}\delta + \arg\left(1 + e^{-i\frac{\pi}{2}\delta} \left(\frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta}\right)\right).
\end{aligned}$$

Next, let us consider

$$\begin{aligned}
\arg\left(\frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta}\right) &= \text{Tan}^{-1}\left(\frac{\frac{\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^\delta}}{\frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta}}\right) \\
&= \text{Tan}^{-1}\left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)}\right).
\end{aligned}$$

On the other hand, applying the some method in the result by Nunokawa [2], we have

$$\begin{aligned}
& \left|\frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta} + \frac{i\delta k}{(\beta(\alpha) + 1)(1 - \beta(\alpha))a^\delta}\right| \\
&> \left|\frac{\delta}{2(1 - \beta(\alpha)^2)a^\delta} \left(a + \frac{1}{a}\right)\right| \\
&= \frac{\delta}{2(1 - \beta(\alpha))a^\delta} \left(a^{1-\delta} + \frac{1}{a^{1+\delta}}\right) \\
&\geq \frac{\delta}{2(1 - \beta(\alpha)^2)} \left(\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1-\delta}{2}} + \left(\frac{1+\delta}{1-\delta}\right)^{\frac{1+\delta}{2}}\right) \\
&= R_0
\end{aligned} \tag{3}$$

say.

Therefore, for the case

$$\arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2}\delta$$

and

$$\text{Tan}^{-1}\left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)}\right) \geq \frac{\pi}{2}\delta,$$

we have

$$\begin{aligned}
& \arg \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right) \\
&= \arg(p(z_0) - \beta(\alpha)) + \arg \left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \frac{1}{p(z_0)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)} \right) \\
&\geq \frac{\pi}{2} \delta + \arg \left(1 + R_0 e^{i(\Theta - \frac{\pi}{2} \delta)} \right) \\
&= \frac{\pi}{2} \delta + \text{Tan}^{-1} \frac{R_0 \sin \left(\Theta - \frac{\pi}{2} \delta \right)}{1 + R_0 \cos \left(\Theta - \frac{\pi}{2} \delta \right)},
\end{aligned}$$

where

$$\Theta = \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) + 1)(\beta(\alpha) - \alpha)}.$$

This is the contradiction for the condition of the theorem.

On the other hand, for the case

$$\arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2} \delta$$

and

$$\Theta = \text{Tan}^{-1} \left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)} \right) < \frac{\pi}{2} \delta,$$

putting $a \rightarrow +0$ in (3), we easily have

$$\begin{aligned}
& \arg \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right) \\
&= \arg(p(z_0) - \beta(\alpha)) + \arg \left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \frac{1}{p(z_0)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)} \right) \\
&\geq \frac{\pi}{2} \delta + \arg e^{i(\Theta - \frac{\pi}{2} \delta)} \\
&= \Theta \\
&= \text{Tan}^{-1} \frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)}.
\end{aligned}$$

This is also the contradiction for the theorem.

For the case

$$\arg(p(z_0) - \beta(\alpha)) = -\frac{\pi}{2} \delta$$

and

$$\text{Tan}^{-1} \left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)} \right) \leq -\frac{\pi}{2} \delta,$$

where

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right)$$

and

$$\left(\frac{p(z_0) - \beta(\alpha)}{1 - \beta(\alpha)} \right)^{\frac{1}{\delta}} = -ia \quad (a > 0),$$

or for the case

$$\arg(p(z_0) - \beta(\alpha)) = -\frac{\pi}{2}\delta$$

and

$$\tan^{-1} \left(\frac{\delta k}{(\beta(\alpha) - \alpha)(\beta(\alpha) + 1)} \right) < -\frac{\pi}{2}\delta,$$

where

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right)$$

and

$$\left(\frac{p(z_0) - \beta(\alpha)}{1 - \beta(\alpha)} \right)^{\frac{1}{\delta}} = -ia \quad (a > 0),$$

applying the same method as the above, we have the contradiction for the theorem. Therefore the proof of the theorem is completed. \square

Remark 1. We have to say that Theorem 1 is not sharp. To consider the sharp result, we need another method. Therefore we leave this problem for our discussion.

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Mamoru Nunokawa
 Emeritus Professor
 University of Gunma
 Hoshikuki-Cho, 798-8
 Chuou-Ward, Chiba 260-0808
 Japan
 E-mail : mamoru_nuno@doctor.nifty.jp

Shigeyoshi Owa and Hitoshi Shiraishi
 Department of Mathematics
 Kinki University
 Higashi-Osaka, Osaka 577-8502
 Japan
 E-mail : owa@math.kindai.ac.jp
 shiraishi@math.kindai.ac.jp